9.10 Problem-Solving Strategies

In this Chapter, we have seen how Biot-Savart and Ampere's laws can be used to calculate magnetic field due to a current source.

9.10.1 Biot-Savart Law:

The law states that the magnetic field at a point *P* due to a length element $d\vec{\mathbf{s}}$ carrying a steady current *I* located at $\vec{\mathbf{r}}$ away is given by

$$d\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}}\times\vec{\mathbf{r}}}{r^3}$$

The calculation of the magnetic field may be carried out as follows:

(1) Source point: Choose an appropriate coordinate system and write down an expression for the differential current element $I\,d\vec{\mathbf{s}}$, and the vector $\vec{\mathbf{r}}\,'$ describing the position of $I\,d\vec{\mathbf{s}}$. The magnitude $r' = |\vec{\mathbf{r}}\,'|$ is the distance between $I\,d\vec{\mathbf{s}}$ and the origin. Variables with a "prime" are used for the source point.

(2) Field point: The field point *P* is the point in space where the magnetic field due to the current distribution is to be calculated. Using the same coordinate system, write down the position vector $\vec{\mathbf{r}}_P$ for the field point *P*. The quantity $r_P = |\vec{\mathbf{r}}_P|$ is the distance between the origin and *P*.

(3) Relative position vector: The relative position between the source point and the field point is characterized by the relative position vector $\vec{\mathbf{r}} = \vec{\mathbf{r}}_P - \vec{\mathbf{r}}\,'$. The corresponding unit vector is

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r} = \frac{\vec{\mathbf{r}}_P - \vec{\mathbf{r}}\,'}{|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}\,'|}$$

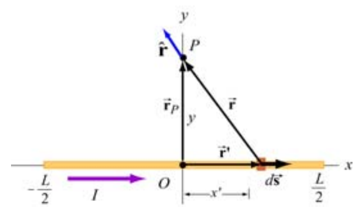
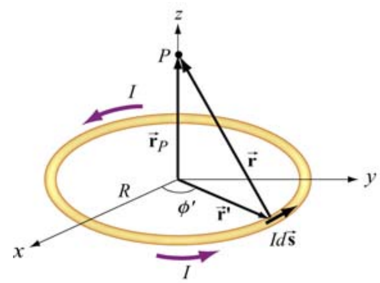
where $r = |\vec{\mathbf{r}}| = |\vec{\mathbf{r}}_P - \vec{\mathbf{r}}\,'|$ is the distance between the source and the field point *P*.

(4) Calculate the cross product $d\vec{\mathbf{s}}\times\hat{\mathbf{r}}$ or $d\vec{\mathbf{s}}\times\vec{\mathbf{r}}$. The resultant vector gives the direction of the magnetic field $\vec{\mathbf{B}}$, according to the Biot-Savart law.

(5) Substitute the expressions obtained to $d\vec{\mathbf{B}}$ and simplify as much as possible.

(6) Complete the integration to obtain $\vec{\mathbf{B}}$ if possible. The size or the geometry of the system is reflected in the integration limits. Change of variables sometimes may help to complete the integration.

Below we illustrate how these steps are executed for a current-carrying wire of length L and a loop of radius R.

Current distribution	Finite wire of length L	Circular loop of radius R
Figure		
(1) Source point	$\vec{\mathbf{r}}' = x'\hat{\mathbf{i}}$ $d\vec{\mathbf{s}} = (d\vec{\mathbf{r}}'/dx')dx' = dx'\hat{\mathbf{i}}$	$\vec{\mathbf{r}}' = R(\cos\phi'\hat{\mathbf{i}} + \sin\phi'\hat{\mathbf{j}})$ $d\vec{\mathbf{s}} = (d\vec{\mathbf{r}}'/d\phi')d\phi' = Rd\phi'(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}})$
(2) Field point P	$\vec{\mathbf{r}}_P = y\hat{\mathbf{j}}$	$\vec{\mathbf{r}}_P = z\hat{\mathbf{k}}$
(3) Relative position vector $\vec{\mathbf{r}} = \vec{\mathbf{r}}_P - \vec{\mathbf{r}}'$	$\vec{\mathbf{r}} = y\hat{\mathbf{j}} - x'\hat{\mathbf{i}}$ $r = \lvert\vec{\mathbf{r}}\rvert = \sqrt{x'^2 + y^2}$ $\hat{\mathbf{r}} = \dfrac{y\hat{\mathbf{j}} - x'\hat{\mathbf{i}}}{\sqrt{x'^2 + y^2}}$	$\vec{\mathbf{r}} = -R\cos\phi'\hat{\mathbf{i}} - R\sin\phi'\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ $r = \lvert\vec{\mathbf{r}}\rvert = \sqrt{R^2 + z^2}$ $\hat{\mathbf{r}} = \dfrac{-R\cos\phi'\hat{\mathbf{i}} - R\sin\phi'\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{\sqrt{R^2 + z^2}}$
(4) The cross product $d\vec{\mathbf{s}} \times \hat{\mathbf{r}}$	$d\vec{\mathbf{s}} \times \hat{\mathbf{r}} = \dfrac{y\,dx'\hat{\mathbf{k}}}{\sqrt{y^2 + x'^2}}$	$d\vec{\mathbf{s}} \times \hat{\mathbf{r}} = \dfrac{R\,d\phi'(z\cos\phi'\hat{\mathbf{i}} + z\sin\phi'\hat{\mathbf{j}} + R\hat{\mathbf{k}})}{\sqrt{R^2 + z^2}}$
(5) Rewrite $d\vec{\mathbf{B}}$	$d\vec{\mathbf{B}} = \dfrac{\mu_0 I}{4\pi}\dfrac{y\,dx'\hat{\mathbf{k}}}{(y^2 + x'^2)^{3/2}}$	$d\vec{\mathbf{B}} = \dfrac{\mu_0 I}{4\pi}\dfrac{R\,d\phi'(z\cos\phi'\hat{\mathbf{i}} + z\sin\phi'\hat{\mathbf{j}} + R\hat{\mathbf{k}})}{(R^2 + z^2)^{3/2}}$
(6) Integrate to get $\vec{\mathbf{B}}$	$B_x = 0$ $B_y = 0$ $B_z = \dfrac{\mu_0 I y}{4\pi}\displaystyle\int_{-L/2}^{L/2}\dfrac{dx'}{(y^2 + x'^2)^{3/2}}$ $= \dfrac{\mu_0 I}{4\pi}\dfrac{L}{y\sqrt{y^2 + (L/2)^2}}$	$B_x = \dfrac{\mu_0 I R z}{4\pi(R^2 + z^2)^{3/2}}\displaystyle\int_0^{2\pi}\cos\phi'\,d\phi' = 0$ $B_y = \dfrac{\mu_0 I R z}{4\pi(R^2 + z^2)^{3/2}}\displaystyle\int_0^{2\pi}\sin\phi'\,d\phi' = 0$ $B_z = \dfrac{\mu_0 I R^2}{4\pi(R^2 + z^2)^{3/2}}\displaystyle\int_0^{2\pi}d\phi' = \dfrac{\mu_0 I R^2}{2(R^2 + z^2)^{3/2}}$

9.10.2 Ampere's law:

Ampere's law states that the line integral of $\vec{\mathbf{B}}\cdot d\vec{\mathbf{s}}$ around any closed loop is proportional to the total current passing through any surface that is bounded by the closed loop:

$$\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} = \mu_0 I_{\text{enc}}$$

To apply Ampere's law to calculate the magnetic field, we use the following procedure:

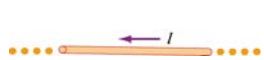
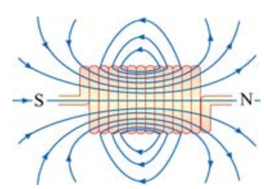
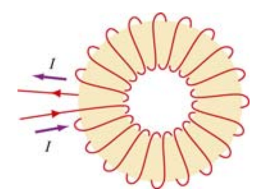
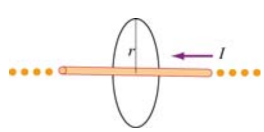
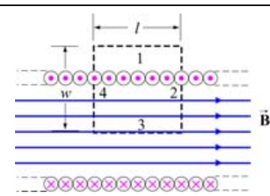
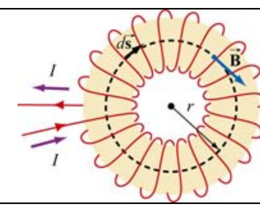
(1) Draw an Amperian loop using symmetry arguments.

(2) Find the current enclosed by the Amperian loop.

(3) Calculate the line integral $\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}}$ around the closed loop.

(4) Equate $\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}}$ with $\mu_0 I_{\text{enc}}$ and solve for $\vec{\mathbf{B}}$.

Below we summarize how the methodology can be applied to calculate the magnetic field for an infinite wire, an ideal solenoid and a toroid.

System	Infinite wire	Ideal solenoid	Toroid
Figure			
(1) Draw the Amperian loop			
(2) Find the current enclosed by the Amperian loop	$I_{\text{enc}} = I$	$I_{\text{enc}} = NI$	$I_{\text{enc}} = NI$
(3) Calculate $\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}}$ along the loop	$\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} = B(2\pi r)$	$\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} = Bl$	$\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} = B(2\pi r)$
(4) Equate $\mu_0 I_{\text{enc}}$ with $\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}}$ to obtain $\vec{\mathbf{B}}$	$B = \dfrac{\mu_0 I}{2\pi r}$	$B = \dfrac{\mu_0 NI}{l} = \mu_0 nI$	$B = \dfrac{\mu_0 NI}{2\pi r}$